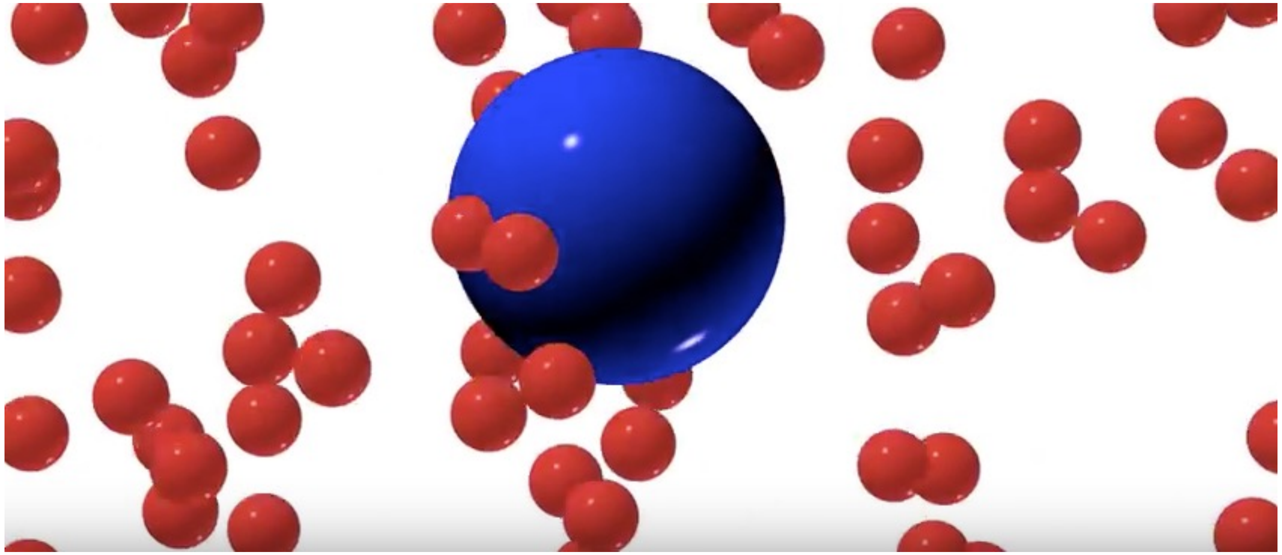
The Physics of Energy

Luca Gammaitoni

Corso di Laurea in Fisica

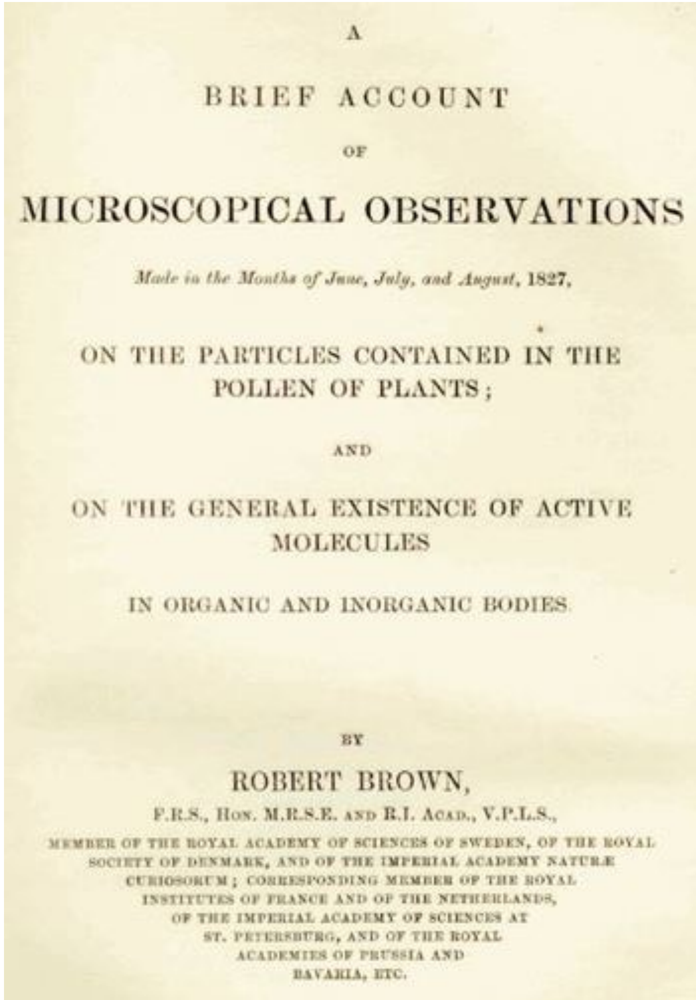
Brownian motion



Per la fenomenologia, vedere: https://www.youtube.com/watch?v=cVkRnEpbeaI
https://www.youtube.com/watch?v=R4t32aGtO3c

Robert Brown

(Montrose, 21 dicembre 1773 – Londra, 10 giugno 1858)



A

BRIEF ACCOUNT

OF

MICROSCOPICAL OBSERVATIONS

Made in the Months of June, July, and August, 1827,

ON THE PARTICLES CONTAINED IN THE POLLEN OF PLANTS;

AND

ON THE GENERAL EXISTENCE OF ACTIVE MOLECULES

IN ORGANIC AND INORGANIC BODIES

BY

ROBERT BROWN,

F.R.S., HON. M.R.S.E. AND R.I. ACAD., V.P.L.S.,

MEMBER OF THE ROYAL ACADEMY OF SCIENCES OF SWEDEN, OF THE ROYAL SOCIETY OF DENMARK, AND OF THE IMPERIAL ACADEMY NATURÆ CURIOSORUM; CORRESPONDING MEMBER OF THE ROYAL INSTITUTES OF FRANCE AND OF THE NETHERLANDS, OF THE IMPERIAL ACADEMY OF SCIENCES AT ST. PETERSBURG, AND OF THE ROYAL ACADEMIES OF PRUSSIA AND BAVARIA, ETC.

Jan Ingenhousz described the irregular motion of coal dust particles on the surface of alcohol in 1785. However, the discovery of this phenomenon is often credited to the botanist Robert Brown in 1827.

Sviluppi

. . . il moto browniano ci fornisce una delle più belle e dirette dimostrazioni sperimentali dei fondamentali principi della teoria meccanica del calore, manifestando quell'assiduo stato vibratorio che esser deve e nei liquidi e nei solidi ancor quando non si muta in essi la temperatura.

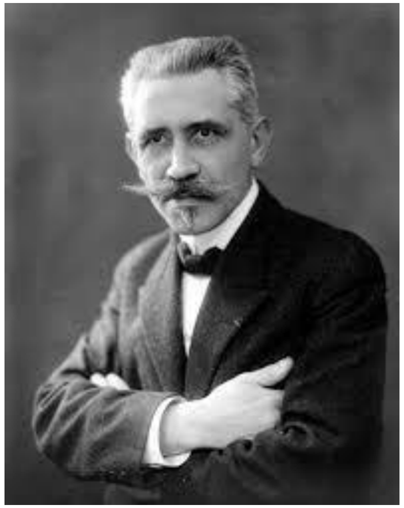
Giovanni Cantoni

(Su alcune condizioni fisiche dell'affinità,e sul moto browniano, in Rendiconti del Regio istituto lombardo di scienze e lettere, s. 2, I (1868), pp. 56-57)

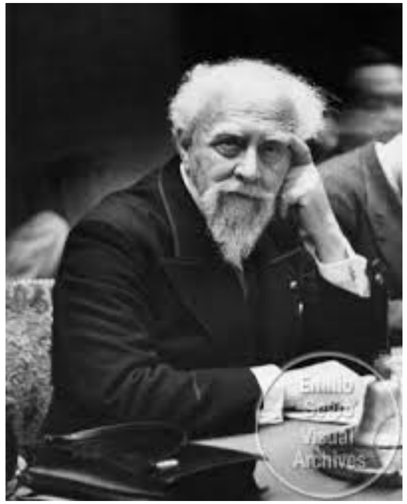
I protagonisti

Albert Einstein
(1879 - 1955)

Marian Smoluchowski
(1872 - 1917)



Paul Langevin
(1872 - 1946)



Jean Baptiste Perrin
(1870 - 1942)

Timeline

J. Ingenhousz 1795
R. Brown 1827

A. Einstein 1905
M. Smoluchowski 1906

J. Perrin 1908
P. Langevin 1908

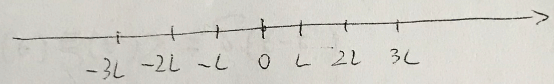
Moto Browniano

Robert Brown 1827 (J. Ingenhousz 1785)

A. Einstein 1905 – M. Smoluchowski (1906)

Analogia con il cosiddetto RANDOM WALK

Consideriamo il random walk semplice:
- 1 dimensione
- passi di uguale ampiezza L
- 50% probabilità di avere $+L$ e $-L$.



Domande:
- dove si trova in media dopo n passi? $\langle x_n \rangle = 0$
- quanto si allontana in media $\langle x_n^2 \rangle$? $\langle x_n^2 \rangle = nL^2$
  (la varianza della distanza è $\sigma^2_{x_n}$)

$$X_n = X_{n-1} \pm L \Rightarrow \langle x_n^2 \rangle = \langle (x_{n-1} \pm L)^2 \rangle =$$

$$= \langle x_{n-1}^2 + L^2 \pm 2 x_{n-1} L \rangle =$$

$$= \langle x_{n-1}^2 \rangle + L^2 \pm \langle x_{n-1} \rangle L =$$

($\langle x_{n-1} \rangle = 0$)

$$= \langle x_{n-1}^2 \rangle + L^2$$

Ora se $n = 1$ e $X_0 = 0$ si ha $\langle x_1^2 \rangle = 0 + L^2 = L^2$

$$\langle x_2^2 \rangle = \langle x_1^2 \rangle + L^2 = 2L^2$$

$$\dots$$

$$\langle x_n^2 \rangle = nL^2$$

Ovvero la varianza cresce con t

Torniamo ora al problema del moto Browniano: particelle di massa m immerse nel liquido.

Seguiremo la derivazione dovuta a Langevin (per noi più semplice di quella di Einstein).

Equazione del moto di ~~Einstein~~ Newton (1 dim)

(1) $$m\ddot{x} = f \qquad f = -\gamma\dot{x} + \xi(t)$$

$$\langle \xi(t) \rangle = 0$$

$$\langle \xi(t)\,\xi(t') \rangle = \delta(t-t')$$

$-\gamma\dot{x}$: forza di attrito viscoso

$\xi(t)$: forza stocastica

$$\gamma = 6\pi r \eta \qquad \text{legge di Stokes}$$

r: raggio della sfera

η: coefficiente di viscosità

La (1) diventa

$$m\ddot{x} = -\gamma\dot{x} + \xi(t)$$

Moltiplico tutto per x

(2) $$m x\ddot{x} = -\gamma x\dot{x} + x\,\xi(t)$$

osservo che $$\frac{d}{dt}x^2 = 2x\dot{x}$$

quindi $$\frac{d^2}{dt^2}x^2 = \frac{d}{dt}(2x\dot{x}) = 2x\ddot{x} + 2\dot{x}^2$$

da cui $x\ddot{x} = \frac{1}{2}\frac{d^2}{dt^2}x^2 - \dot{x}^2$ e $x\dot{x} = \frac{1}{2}\frac{d}{dt}x^2$

sostituisco nella (1)

$$\frac{m}{2}\frac{d^2}{dt^2}x^2 - m\dot{x}^2 = -\gamma\frac{1}{2}\frac{d}{dt}x^2 + x\,\xi(t)$$

prendo il valor medio (integrale e media commutano)

$$\frac{m}{2}\frac{d^2}{dt^2}\langle x^2\rangle - m\langle \dot{x}^2\rangle = -\gamma\frac{1}{2}\frac{d}{dt}\langle x^2\rangle + \langle x\,\xi(t)\rangle$$

ove $\langle x\,\xi(t)\rangle = 0$ perché non correlati

$$\frac{1}{2}m\langle \dot{x}^2\rangle = \frac{1}{2}KT$$ (Equip. dell'energia)

ipotesi dell'eq. termico

quindi

$$\frac{m}{2}\frac{d^2}{dt}\langle x^2\rangle - KT = -\gamma\frac{1}{2}\frac{d}{dt}\langle x^2\rangle$$

Chiamo $\alpha = \frac{d}{dt}\langle x^2\rangle$

$$\frac{m}{2}\dot{\alpha} - KT + \gamma\frac{1}{2}\alpha = 0$$

$$\frac{1}{2}m\dot{\alpha} + \gamma\frac{1}{2}\alpha - KT = 0$$

$$\dot{\alpha} + \frac{\gamma}{m}\alpha - \frac{2KT}{m} = 0$$

Eq. differenziale lineare e coeff. costanti

$$\dot{\alpha} + \frac{\gamma}{m}\alpha - \frac{2KT}{m} = 0$$

$$\alpha = a + b\, e^{-ct}$$

$$\dot{\alpha} = -bc\, e^{-ct}$$

sostituisco $-bc\,e^{-ct} + \frac{\gamma}{m} a + \frac{\gamma}{m} b e^{-ct} - \frac{2KT}{m} = 0$

che comporta $+\frac{\gamma}{m} a = \frac{2KT}{m} \Rightarrow a = \frac{2KT}{\gamma}$

$c = \frac{\gamma}{m}$

$b =$ cost. qualunque
fissata dal valore a $t=0$

quindi $\alpha = \frac{2KT}{\gamma} + C\, e^{-\frac{\gamma}{m}t}$

↖ transiente

dopo un tempo t_0 (grande quanto ?)

$$\alpha = \frac{2KT}{\gamma} \Rightarrow \alpha = \frac{d}{dt}\langle x^2 \rangle = \frac{2KT}{\gamma}$$

da cui $\langle x^2 \rangle = \frac{2KT}{\gamma} t$

↖ Rem!

Nel caso tridimensionale

$$\langle x^2 \rangle = 3 \cdot 2 \frac{KT}{\gamma} t = \frac{6KT}{\gamma} t$$

Se sostituisco $\gamma = 6\pi r\eta \Rightarrow \langle x^2 \rangle = \frac{KT}{\pi r \eta} t$ (Einstein)

To learn more:

Probabilità in Fisica

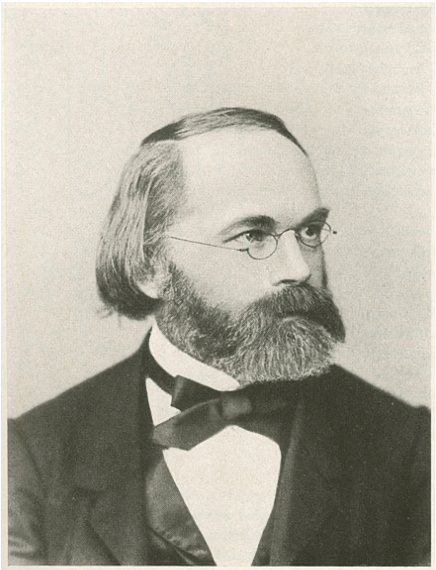
Guido Boffetta, Angelo Vulpiani

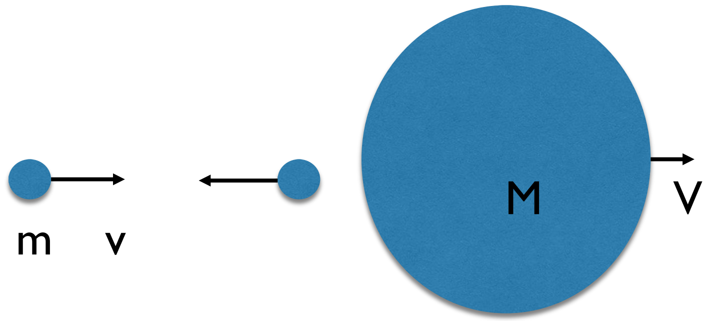
Chap. 4 Il moto Browniano: primo incontro con i processi stocastici

Peculiarities in the treatment of stochastic processes: the Brownian motion misinterpreted

Before Einstein 1905 model of the Brownian motion, other attempts have been made, but the atomistic interpretation was erroneously discarded.

1879 Carl Wilhelm von Nägeli (26 March 1817 – 10 May 1891) Swiss botanist





Momentum acquired by the pollen grain: $M\,V = 2\,mv$

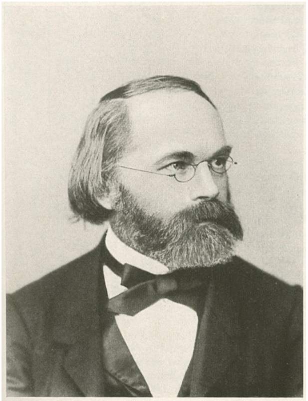
$$V = 2\,\frac{m}{M}\,v \quad \text{but} \quad \frac{m}{M} = \left(\frac{r}{R}\right)^3 = \left(\frac{5\;10^{-10}}{10^{-6}}\right)^3 \sim 10^{-10}$$

thus $V \sim 10^{-10} v$ Due to the equipartition, we have $v = \sqrt{3\,\dfrac{KT}{m}} \sim 6\;10^{2}\;\; m/s$

and thus $V \sim 6\;10^{-8}\;\; m/s$

This is roughly 2 orders of magnitude smaller than the observed velocity!

The stochastic force is not the sum of independent kicks



On the base of a significant disagreement with the observations, von Nägeli concluded that the kinetic theory model did not work.

The stochastic force is often **misrepresented** as the sum of independent kicks, half favourable, half unfavourable. This is not the case.
We should think at it as the results of a large number N kicks that happen in a very short time. The effectiveness of this force goes roughly as the sqrt(N), where N is the number of kicks per second.

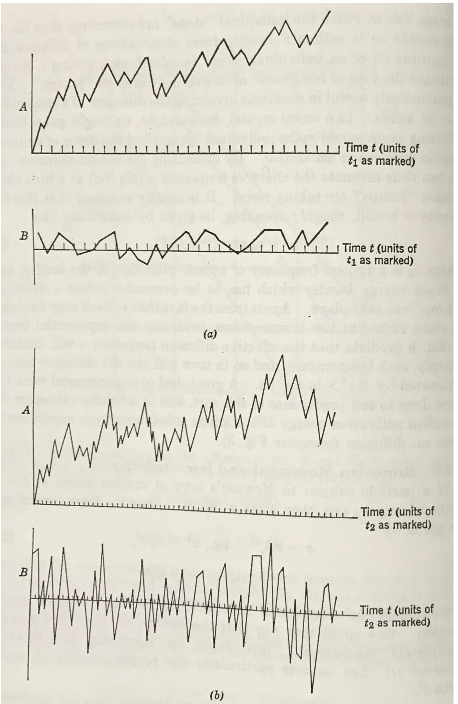
This is the result of a «vision» that takes into account a time scale separation: fast (molecular motion) vs (slow) the pollen grain motion.

Another common mistake: mind the observation time

If we calculate the standard deviation of the velocity of the pollen grain using the kinetic theory we have:

$\frac{1}{2}M\langle V^2\rangle = \frac{3}{2}KT$ which gives $\sqrt{\langle V^2\rangle} \sim 2\ 10^{-3}$ m/s orders of magnitude larger than the observed one.

Why ?



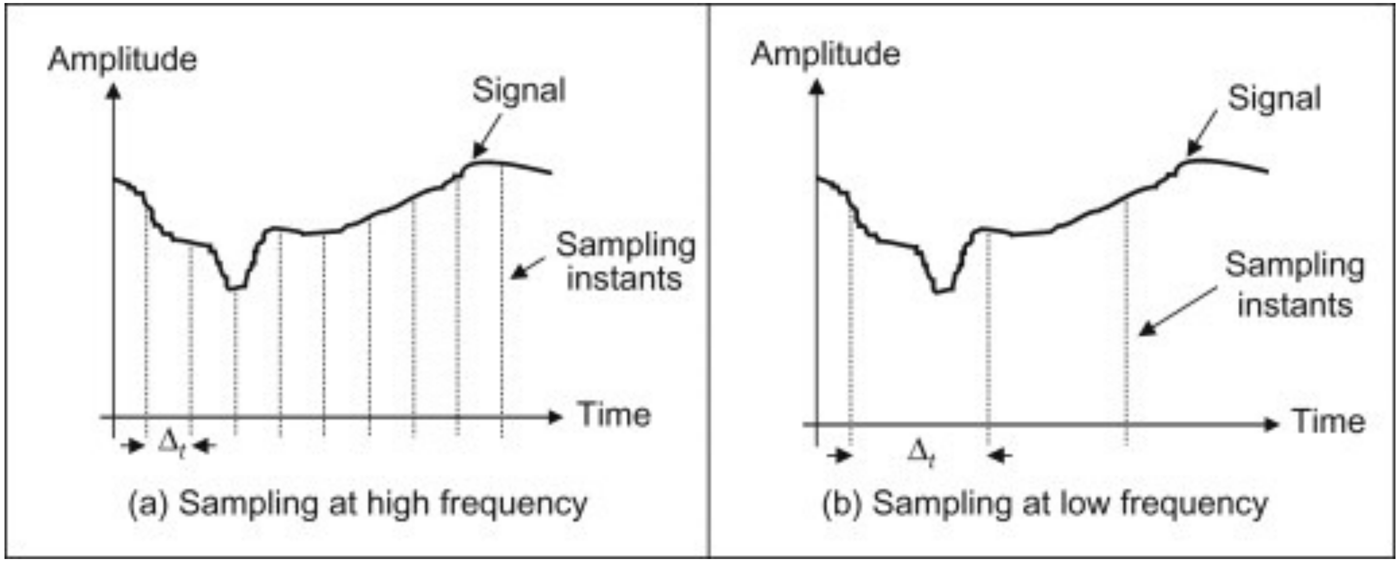
Apparently the amplitude of the velocity in this process depends on the sampling time.
Why?

The Nyquist–Shannon sampling theorem

Is a theorem in the field of digital signal processing which serves as a fundamental bridge between continuous-time signals and discrete-time signals. It establishes a sufficient condition for a sample rate that permits a discrete sequence of samples to capture all the information from a continuous-time signal of finite bandwidth.

Time discretization

If we have a continuous signal x(t) we can represent it by means of a series of time discret samples, each separated by a time distance ΔT: x(t0 + i ΔT) for 1= 1…n.



The Nyquist–Shannon sampling theorem

If a function x(t) contains no frequencies higher than B hertz, it is completely determined by giving its ordinates at a series of points spaced 1/(2B) seconds apart.

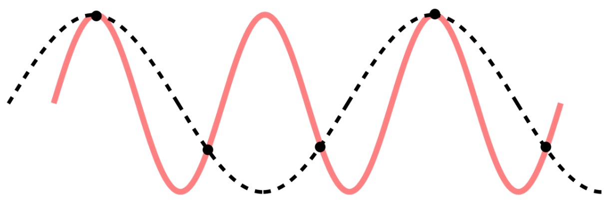
$$\Delta T < \frac{1}{2B}$$

From the sampled values we can reconstruct the continuous function as follows:

$$x(t) = \sum_{n=-\infty}^{\infty} x(nT) \cdot \mathrm{sinc}\left(\frac{t - nT}{T}\right) \qquad \text{with} \qquad \mathrm{sinc}(x) = \frac{\sin x}{x}$$

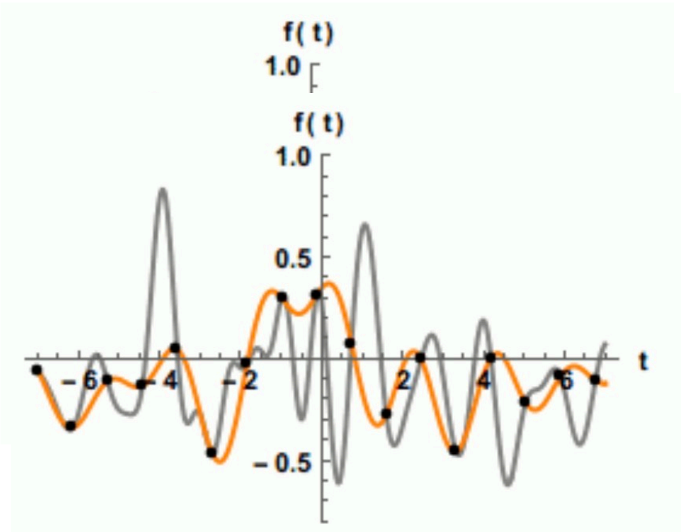
Whittaker–Shannon interpolation formula

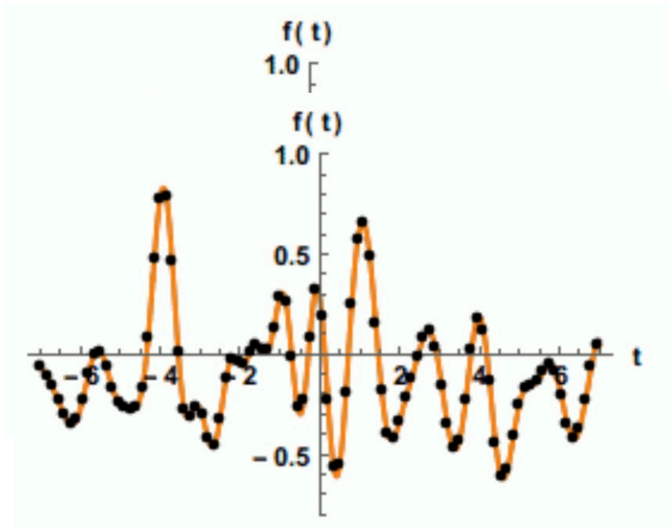
If we do not respect the prescription of the Nyquist–Shannon sampling theorem, we have an error called **aliasing**.



The reconstructed signal is different from the original one and, specifically has amplitude in lower frequency regions with respect to the original.

Aliasing error





Undersampled signal

$$\Delta T > \frac{1}{2B}$$

Properly sampled signal

$$\Delta T < \frac{1}{2B}$$

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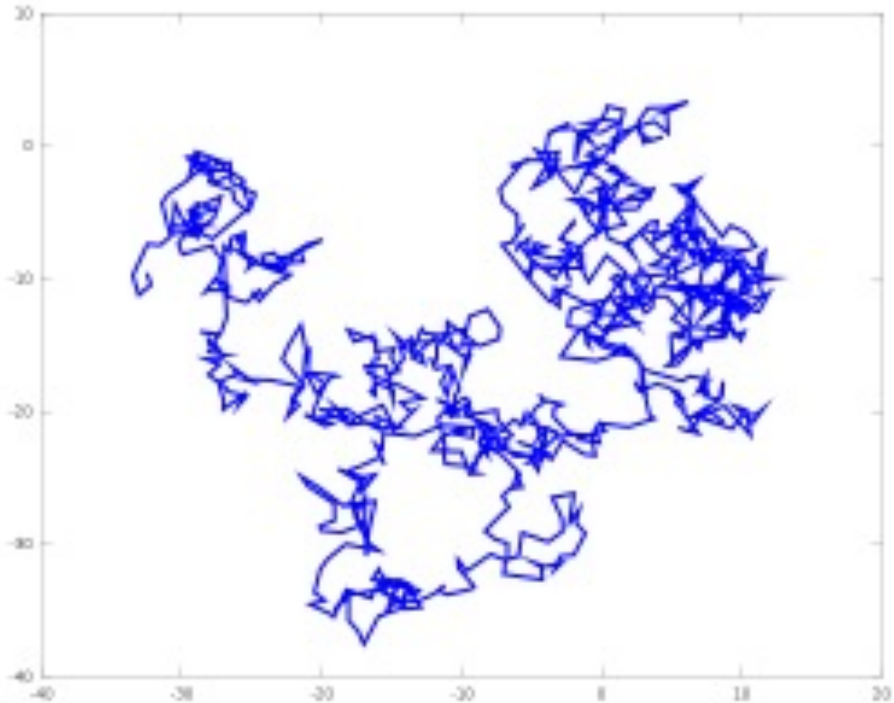

Rem: mind the sampling interval